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SOME INVESTIGATIONS RELATING TO THE ELASTOSTATICS OF A TAPERED TUBE

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Separation of variables for the harmonic and biharmonic is not possible for the coordinate system describing a tapered hollow cylinder so alternative methods are required. The method of characteristics, Poisson integral representations, series in two coordinate variables, and the method of successive approximations are discussed in some detail. The only method to show promise for future work is the method of successive approximations used by A. Zak to investigate singularity stresses at the end of a cylinder.

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TABLE OF CONTENTS

Some Investigations Relating to

the Elastostatics of a Tapered Tube

by

Barry Bernstein

1. Introduction

The problem of elastostatics of a tapered tube is one for which one searches the literature in vain. The problem seems to be close to that of a cylindrical tube, but this appearance is quite deceptive. Several singularities appear in the tapered tube which do not in the cylindrical tube. Furthermore, in a coordinate system appropriate to the cylindrical tube, separation of variables is possible. Not so with the tapered tube. In this report we shall explore some approaches to the problem. No approach that we shall present has at this time shown itself to be the clear way to proceed. However, some of them may have some promise.

After an introduction to the problem, we shall discuss some methods found in the literature. Then we shall present some new exploratory results. Although we cannot be conclusive at this time, we hope that we have opened some possibilities for future development.

References are in the annotated bibliography, Section 8.

Special List of Symbols:

Because we are quoting from different sources which use the same symbols in different ways, and since we wish, with only, perhaps, reasonable modification that the reader be able to recognize the symbols in the quoted sources, we cannot be completely consistent in using a symbol in only one way in this report. For this reason, we have compiled a list of symbols here with the different uses of the same symbol explained. Symbols are listed roughly in the order in which they appear in the text, except that all listings of different uses of the same symbol appear together. If the reader will refer to this list, confusion will be avoided.

Symbol	Uses
V	potential function (section 2)
V	a solution of (28) with m=1 (section 7)
C_0, C_1, C_2, C_3	constants (section 2)
V^m	Fourier component of potential (section 2)
R, Θ, Z	cylindrical coordinates
m	an integer
U^m	reduced potential (section 2)
∇^2	defined by equation (4)
$\zeta = \xi + i\eta$	coordinates for the tapered tube (section 2) (essentially ξ and η of section 2 are the Niven coordinates ρ and θ of section 5)
ξ, η	characteristic coordinates (sections 5,6)
c	the distance of the intersection of the inner and outer surface of the tapered tube from the axis
T, u, v	separation functions (equation 6)
u	some arbitrary function (section 6)
ρ	the distance from the singularity in Zak's coordinate system (section 3) except that Zak takes $c = 1$, this is the same as the Niven coordinate r of section 5

ρ	a Niven coordinate (sections 5,6,7) in which $\rho = \ell n\, r$
ϕ	Zak's angular coordinate (section 3). The same as the Niven coordinate θ (sections 5,6,7)
Ω, Γ	Southwell potentials (section 3)
Γ	gamma function (section 4)
Γ	a curve (section 6)
$F_p(\phi)$	a separation function (section 3)
λ	a constant (section 3)
α	the taper angle (see figure)
f	a function (equation 7)
x,y,z	Cartesian coordinates
z	defined by equation (23) (section 6)
Φ_n	same as V^m, but in Hein's notation
Φ_n^s, Φ_n^c	self explanatory - (equation 9)
$r, \tilde{\theta}, \phi$	Niven coordinates (section 5)
θ	$\pi/2 - \tilde{\theta}$
v,t	some special coordinates (equation 15 and following equation)
A_n	some constant coefficients (equation 17)
W	defined by equation (18)
H	characteristic function (section 6)
ρ_0, θ_0	Niven coordinates of a given point (section 6)
ξ_0, η_0	characteristic coordinates of a given point (section 6)
$F(\alpha, \beta, \gamma, z)$	hypergeometric function (section 6)
P_1, P_2	points of intersection of characteristics with curve
X,Y	as defined in the equation following (25)

Q	as defined in equation (27)
U	as in equation (28) - same as U^m
$f_k(z)$	functions to be determined (equation 29)
p	an index (section 7)
c_j	constants to be determined (section 7)
n	an integer (equation 34)
$h(n)$	the greatest integer in $n/2$ (section 7)
b_j	coefficients to be determined (equation 39)

2. The Nature of the Problem

That the problem of elastostatics hangs on the study of Laplace's equation is well known. A review of solutions of such problems in terms of potential functions is given by Green and Zerna, section (5,6) [13]. If one could handle Laplace's equation for the tapered tube, then elastostatic problems would be accessible.

The first effort, then, that seems reasonable is to see if separation of variables is possible. We turn, then, to the work by Snow [1]. We consider here chapter IX, p. 228 of this work.

In cylindrical coordinates R, Z, Θ, we have for Laplace's equation for a potential V

$$\frac{\partial^2 V}{\partial R^2} + \frac{\partial^2 V}{\partial Z^2} + \frac{1}{R}\frac{\partial V}{\partial R} + \frac{1}{R^2}\frac{\partial^2 V}{\partial \Theta^2} = 0 . \tag{1}$$

From equation (1), we may immediately separate out the angular coordinate Θ by writing V in a Fourier series in Θ. Indeed, Snow writes

$$V = C_0 + C_1 Z + (C_2 + C_3 Z)\,\ell n\, R + \sum_{m=0}^{\infty} V^m(R,Z)\cos m\,(\Theta - \Theta_m)$$

where C_0, C_1, C_2, C_3 and Θ_m are constants. The coefficient V^m satisfies

$$\frac{\partial^2 V^m}{\partial Z^2} + \frac{\partial^2 V^m}{\partial R^2} + \frac{1}{R}\frac{\partial V^m}{\partial R} - \frac{m^2 V^m}{R^2} = 0. \tag{2}$$

Or, putting

$$V^m = R^{-\frac{1}{2}}\, U^m$$

in equation (2) one obtains an equation for the reduced potential U^m, namely

$$\nabla^2 U^m + \frac{\frac{1}{4} - m^2}{R^2}\, U^m = 0 \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\,. \tag{4}$$

Now if one looks at the R-Z plane one sees that the trace on this plane of a tapered region can be represented as a wedge, one side of which is parallel to the Z-axis at some distance, say, c, from the axis, and crossing the other side at, say, the Z-axis at an angle α (see figure), which we call the taper angle. A conformal mapping, then, from the $Z + iR$ plane into the plane of $\zeta = \xi + i\eta$ given by

$$\zeta = \log(Z + R - iC)$$

gives for (3)

$$\nabla^2 U^m + \frac{e^{2\xi}}{(c + e^{\xi}\sin\eta)^2}\left(\frac{1}{4} - m^2\right) U^m = 0. \qquad (5)$$

Now equation (5) is an equation in ξ, η, which are natural coordinates for the tapered tube. Indeed, in this coordinate system, the surfaces of the tube become $\zeta = 0$ and $\zeta = \alpha$. One may say more: This is essentially the only orthogonal coordinate system in which the surfaces of the tube become coordinate surfaces for any value of α.

We now ask the question whether or not one may find solutions of the form

$$V^m = T(\xi, \eta)u(\xi)v(\eta) \qquad (6)$$

where T is to be found. The answer is given by Snow (pp. 252-253). It appears that the answer is no, since

$$\frac{\partial^2}{\partial\xi\,\partial\eta}\,\frac{e^{2\xi}}{(a + e^{\xi}\sin\eta)^2} = \frac{6e^{3\xi}\cos\eta\,(1 + e^{\xi}\cos\eta)}{(a + e^{\xi}\sin\eta)^2} \neq 0$$

which, by application of Snow's result to our equation implies that separation of variables, even to within a known factor T, is not possible for the coordinate system (ξ, η).

Techniques of separation of variables, with all their ramifications, then fail. Other techniques must then be sought. And a look at some of these is then our task.

3. The Work of Zak

Here we shall discuss a technique used by Zak for solving a problem of a cylinder with stress singularities. The method happens to involve the Southwell potentials [14], but the essential feature of it is the method of obtaining a sequence of functions which approach a solution.

Zak used a coordinate system which is essentially that developed in the previous section and, indeed, is equivalent to Niven's coordinates (section 5). If we replace ξ by the letter ρ and η by the letter ϕ, we shall have the coordinate system which he uses. In this section we shall adhere to Zak's notation. However, the same letter ρ will be used differently elsewhere in this report, so caution on the part of the reader is urged. Please refer to the list of symbols. If referred to the tapered tube, Zak's coordinates are the Niven coordinates normalized so that $c = 1$.

The Southwell potentials as modified by Zak satisfy the equations

$$\frac{\partial^2\Omega}{\partial R^2} + \frac{1}{R}\frac{\partial\Omega}{\partial R} + \frac{\partial^2\Omega}{\partial z^2} = 0$$

$$\frac{\partial^2\Gamma}{\partial R^2} + \frac{1}{R}\frac{\partial\Gamma}{\partial R} + \frac{\partial^2\Gamma}{\partial z^2} = \frac{\partial^2\Omega}{\partial z^2}$$

in cylindrical coordinates. After expressing these equations in terms of his ρ and ϕ, Zak seeks a solution for, say, Ω in the form

$$\Omega = \sum_{p=0}^{\infty} \rho^{\lambda+p} F_p(\phi)$$

and obtains a sequence of equations

$$F_0'' + \lambda^2 F_0 = 0$$

$$F_p'' + (\lambda+p)^2 F_p - \sin\phi \left\{ \sum_{m=0}^{p-1} (\lambda+m)(\sin\phi)^{p-m-1} F_m \right\}$$

$$- \cos\phi \left\{ \sum_{m=0}^{p-1} (\sin\phi)^{p-m-1} F_m' \right\}$$

so that each function F_p depends on the previous ones. A similar technique is applied to Γ.

It is not difficult to see that Zak's technique could readily be applied to the tapered tube problem: The proper coordinate form and the technique are already developed.

At the time of writing of this report, we feel that Zak's method may be the most promising where it can be applied. It appears to have two disadvantages. Zak expands a term as

$$\frac{1}{1 - \rho \sin\phi} = \sum_{m=0}^{\infty} (\rho \sin\phi)^m$$

which has as its domain of convergence a region near $\rho = 0$. (This region was of interest for the study of a singularity at $\rho = 0$.)

For the tapered tube it may be of interest if one limits oneself to regions where $|\rho \sin\phi| < 1$, but this means that the radial length allowed is limited by the angle of taper. For example, for a taper angle of 2°, the expansion is valid for ρ up to about 28 (distance from singularity about 28 times the quantity C in Niven coordinates) and convergence would probably be slow if ρ were near 28.

Although we do not see how to do it at present, it may be possible to apply Zak's technique to a far away region.

But the trouble at the moment is that as one goes toward larger ρ there are points closer and closer to the surface $\phi = 0$ at which $(1 - \rho \sin\phi)^{-1}$ becomes infinite.

The second disadvantage which may be minor is that one does not deal with a sequence of exact solutions. However, this would not necessarily impair its usefulness where convergence is rapid enough. Nevertheless, in the broad study of the question, a search for exact solutions should be made. If such solutions could be assembled into the solution of a problem, they might or might not provide a better method than that of Zak in some given situation. In sections 6 and 7 we report on a search for such solutions.

4. Integral Equations

The method of integral equations rests on the representation of the solution of Laplace's equation as an integral. A review of such integral representations is given by Temple [7], who contends that the culmination of this work is in Whittaker's result that potential functions which are regular near the origin have representations of the form

$$\int_0^{2\pi} f(z + i\,x\cos\beta + i\,y\sin\beta, \beta)\,d\beta. \tag{7}$$

Basically the method of integral equations consists of setting up equations for the unknown function in an integral expression such as (7). These equations are based on the boundary conditions.

A review of the use of the method of integral equations is given by Heins [2], who makes use of the Poisson integral representation: For a function $\Phi_n(R,Z)$ satisfying

$$\frac{\partial^2 \Phi_n}{\partial R^2} + \frac{1}{R}\frac{\partial \Phi_n}{\partial R} + \frac{\partial^2 \Phi_n}{\partial Z^2} - \frac{n^2 \Phi_n}{R^2} = 0, \tag{8}$$

which is the equation(*) satisfied by a Fourier component Φ_n^c or Φ_n^s of

$$\Phi(R,\phi,Z) = \Phi_0(R,Z) + \sum_{n=1}^{\infty} \Phi_n^c(R;Z)\cos n\phi + \sum_{n=1}^{\infty} \Phi_n^s(R,Z)\sin n\phi \tag{9}$$

of an harmonic function Φ, one obtains

$$U_n(R,Z) = \frac{\Gamma(n+1)}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\int_0^{\pi} U_n(0,z+ir\cos\Psi)\sin^{2n}\Psi\,d\Psi \tag{10}$$

where

$$R^n U_n = \Phi_n. \tag{11}$$

Now the validity of the Poisson Integral Representation (10) hangs upon the regularity of the solution on the Z axis. Indeed the assumption of such regularity is stated explicitly by Heins (p. 789) and the problems solved (e.g. a charged disc, or a lens, with axis along the Z-axis) do not violate

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* Note Equation (8) is the same as equation (2) using Heins' notation instead of Snow's.

this condition. Other work which we have found so far [9, 10, 11, 12] does not seem to violate this condition.

We must caution that we have not at the time of writing of this report fully digested the question of whether regularity on the Z-axis is absolutely crucial to whether or not the problem of the tapered tube is amenable to a Poisson Integral type analysis. However, the Z-axis is outside the domain of required validity of solutions to such problems. So there is no reason to expect that the Poisson Integral will give the answer. On the other hand, neither can one assert at this point that it will not figure in a method of solving the tapered tube problem. Indeed, perhaps we need a solution valid outside the inner surface as well as a solution valid inside the outer surface of the tapered tube, so that their region of common validity will be as desired.

Another method which we feel needs further exploration is that of Snow [1], Chapter IX. Again, we feel at the time of writing of this report that we have not yet seen through the method well enough to be certain that it will apply in whole or in part to the tapered tube. The difficulty at the axis arises in trying to map the R-S plane into the wedge-region which is that of the tapered tube on the R-Z plane without getting into the same type of difficulties with the axis. However, for reasons similar to those stated in connection with the Poisson Integral, we feel that the matter is not at all settled at this time and that we should, indeed, like to consider it further.

Nevertheless, in order to seek integral equation solutions appropriate to the tapered tube, it would be nice to have an integral representation which is tailored to hold in the proper region. To this end, we have carried out an investigation based on the theory of characteristics. It may seem odd to do this today, but in nineteenth century work, the relation of the wave equation to Laplace's equation through the use of complex characteristics was well accepted. We shall present these results as soon as we have discussed the work of Niven.

5. The Work of Niven

A coordinate system appropriate to the tapered tube was treated by Niven [4]. Indeed, he defines a coordinate system r, $\tilde{\theta}$, ϕ by

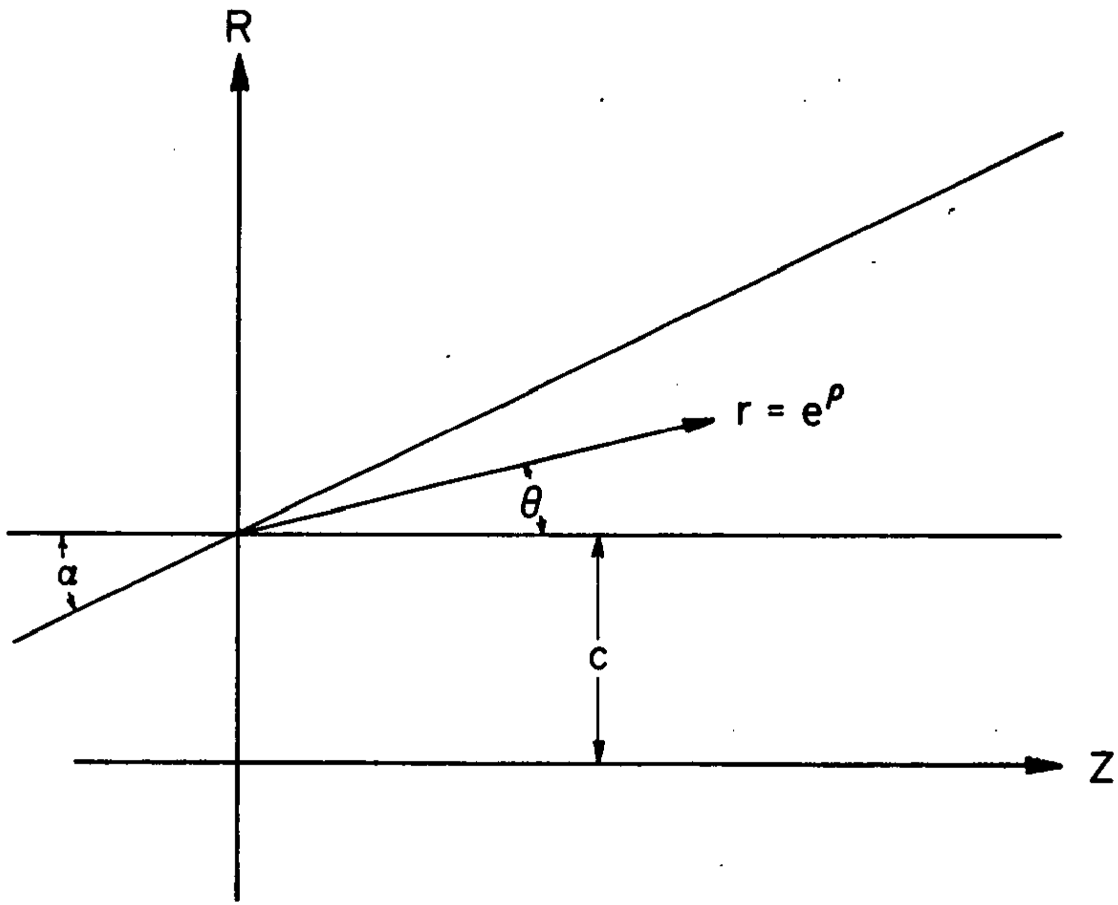
$$x = (C + r\cos\tilde{\theta})\cos\phi$$
$$y = (C + r\cos\tilde{\theta})\sin\phi$$
$$z = r\sin\theta$$

(where the tilda is our notation).

We find it more convenient to deal with the complement of θ. Thus, we shall interchange $\sin\theta$ and $\cos\theta$ to write

$$x = (C + r\sin\theta)\cos\phi$$
$$y = (C + r\sin\theta)\sin\phi$$
$$z = r\cos\theta$$

and, since these differ so trivially from Niven's coordinates, we shall call these Niven coordinates also. It is clear then that in the Niven coordinates the surfaces of the tapered tube with taper angle α are simply $\theta = 0$ and $\theta = \alpha$.



The trace of the outer and inner surfaces of the tapered tube on the R-Z plane. The taper angle is α. The Niven coordinates (r,θ) are shown.

Niven then presented a form of Laplace's equation in these coordinates, namely

$$\frac{\partial}{\partial r} r\,(c + r\cos\tilde{\theta})\,\frac{\partial V}{\partial r} + \frac{1}{r}\,\frac{\partial}{\partial \tilde{\theta}}\,(c + r\cos\tilde{\theta})\,\frac{\partial V}{\partial \tilde{\theta}}$$

$$+ \frac{r}{c + r\cos\tilde{\theta}}\,\frac{\partial^2 V}{\partial \phi^2} = 0 \,.$$

It is clear that by making the substitution $\theta = \pi/2 - \tilde{\theta}$, one gets instead

$$\frac{\partial}{\partial r} r\,(c + r\sin\theta)\,\frac{\partial V}{\partial r} + \frac{1}{r}\,\frac{\partial}{\partial \theta}(c + r\sin\theta)\,\frac{\partial V}{\partial \theta}$$

$$+ \frac{r}{c + r\sin\theta}\,\frac{\partial^2 V}{\partial \phi^2} = 0.$$

Niven then writes this equation in terms of $\rho = \ln r$ for the case where V is independent of ϕ. We shall write instead the equation in the general case, namely

$$\frac{\partial}{\partial \rho}\,(c + e^{\rho}\sin\theta)\,\frac{\partial V}{\partial \rho} + \frac{\partial}{\partial \theta}\,(c + e^{\rho}\sin\theta)\,\frac{\partial V}{\partial \theta}$$

$$+ \frac{e^{2\rho}}{c + e^{\rho}\sin\theta}\,\frac{\partial^2 V}{\partial \phi^2} = 0.$$

These Niven coordinates have, in fact, been met before in this report and indeed ρ and θ are respectively the ξ and η of section 2. In this section, however, we shall use ξ and η for other coordinates.

Niven writes Laplace's equation for the case where V is independent of ϕ in terms of (characteristic) coordinates

$$\xi = c + e^{\rho + i\tilde{\theta}}, \qquad \eta = c + e^{\rho - i\tilde{\theta}} \tag{13}$$

as

$$2(\xi + \eta) \frac{\partial^2 V}{\partial\xi\partial\eta} + \frac{\partial V}{\partial\xi} + \frac{\partial V}{\partial\eta} = 0. \tag{14}$$

To him this is just a step in a process which in fact does not seem to lead us very far. Indeed the next step is to put

$$\xi + \eta = e^v \tag{15a}$$

$$\xi - \eta = t \tag{15b}$$

and arrive at

$$\frac{\partial^2 V}{\partial v^2} - e^{2v} \frac{\partial^2 V}{\partial t^2} = 0 \tag{15}$$

which does indeed admit separation of variables solutions.

However, the simple observation that

$$v = \log 2R, \qquad t = 2iZ$$

shows that indeed we are merely back in a slightly unfamiliar form to cylindrical coördinates R and Z and that the separation of variables will yield the familiar Bessel functions.

The rest of Niven's results are of mathematical interest, but do not help us with our tapered tube problem. He does indeed obtain some closed form solutions, but they do not have the proper equipotential surfaces appropriate to the tapered tube problem. Indeed, his method is to seek solutions of the form

$$\sum_{n=0}^{\infty} A_n \, e^{m\rho} \cos n\tilde{\theta} \tag{16}$$

where m depends on n in some way, and then cleverly to choose some such dependence which allows the series to yield a solution.

The method is interesting, but it probably lacks the generality which we need in order to obtain enough solutions that we can handle the tapered tube problem. Indeed, in section 7 below we shall present solutions containing types of terms not found in (16).

In closing, we point out that these type of coordinates were used by Riemann [7] to solve a problem of an anchor ring. Although Riemann's paper is very pretty, again it contains no hint how to seek solutions with equipotential surfaces appropriate to the tapered tube.

6. An Integral Representation of the Solution to Laplace's Equation in Niven's Coordinates

Consider, now, the form of the equation for the potential $V^{(m)}(R,Z)$ in polar coordinates (R,Θ) as given by equation (2). We shall obtain formally an integral representation of the solution to this equation in the Niven coordinates (ρ,θ), where

$$R = c + e^{\rho}\sin\theta \tag{17a}$$

$$Z = e^{\rho}\cos\theta. \tag{17b}$$

To begin with, let us put

$$V^{(m)} = R^{m}W, \tag{18}$$

(where, of course, W will also depend on m, but we find it notationally simpler not to write it explicitly). We obtain, then for (2)

$$\nabla^2 W + \frac{2m+1}{R}\frac{\partial W}{\partial R} = 0. \tag{19}$$

Now let us change to coordinates ξ, η, where

$$\xi = R + iZ = C + ie^{\rho - i\theta} \tag{20a}$$

$$\eta = R - iZ = C - ie^{\rho + i\theta}. \tag{20b}$$

(Note that (20a) and (20b) follow from (17) and (13).)

We then obtain for (19)

$$\frac{\partial^2 W}{\partial\xi\partial\eta} + \frac{2m+1}{2}\,\frac{1}{\xi+\eta}\left(\frac{\partial W}{\partial\xi} + \frac{\partial W}{\partial\eta}\right) = 0. \tag{20}$$

Now (20) is in characteristic form, albeit the characteristics are complex, either in the R, Z coordinates or the ρ, θ coordinates. It is clear, then, that the characteristics are given by

$$\rho + i\theta = \text{constant} \tag{21a}$$

and

$$\rho - i\theta = \text{constant}. \tag{21b}$$

Now equation (20) is of the form given in Sommerfeld's book [15], section 11, in relation to a hydrodynamic example treated by Riemann. The method involves a solution to the adjoint equation

$$\frac{\partial^2 H}{\partial\xi\partial\eta} - \frac{2m+1}{2}\,\frac{\partial}{\partial\xi}\left[\frac{1}{\xi+\eta}\left(\frac{\partial H}{\partial\xi} + \frac{\partial H}{\partial\eta}\right)\right] = 0$$

for each point $P_o(\rho_0, \theta_0)$ such that $H = 1$ at P and

$$\frac{\partial H}{\partial\xi} - \frac{2m+1}{2(\xi+\eta)}H = 0$$

and

$$\frac{\partial H}{\partial\eta} - \frac{2m+1}{2(\xi+\eta)}H = 0$$

on the characteristics η = const and ξ = const respectively which pass through ρ_0, θ_0. Riemann solved this problem. The solution for our case is

$$H = \left(\frac{\xi_0 + \eta_0}{\xi + \eta}\right)^{-\frac{2m+1}{2}} F\left(\frac{1-2m}{2}, \frac{1+2m}{2}, 1, z\right) \tag{22}$$

where

$$z = -\frac{(\xi - \xi_0)(\eta - \eta_0)}{(\xi + \eta)(\xi_0 + \eta_0)}, \tag{23}$$

$F(\alpha, \beta, \gamma, z)$ is the hypergeometric function

$$F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \cdots$$

and

$$\xi_0 = c + i\, e^{\rho_0 - i\theta_0} \tag{24a}$$

$$\eta_0 = c - i\, e^{\rho_0 + i\theta_0}. \tag{24b}$$

To obtain an integral form of the solution of (20), then we apply the relation (7) p.(54) of Sommerfeld [15], which we write in the form

$$W(\rho_0, \theta_0) = \int_\Gamma (X d\xi - Y d\eta) + \tfrac{1}{2}(WH)_{P_1} + \tfrac{1}{2}(WH)_{P_2} \tag{25}$$

where Γ is a portion of a curve on which data are given, P_1 and P_2 are the intersections of the curve with the respective characteristics through (ρ_0, θ_0), the integration is taken to be along Γ from P_1 to P_2. (Here we must be careful that P_1 be the point where $\eta = \eta_0$ intersects Γ and P_2 is the point where $\xi = \xi_0$ intersects Γ.) Also

$$X = \frac{1}{2}\left(H\,\frac{\partial W}{\partial \eta} - W\,\frac{\partial H}{\partial \eta}\right) + \frac{2m+1}{2(\xi+\eta)}\,HW$$

$$Y = \frac{1}{2}\left(H\,\frac{\partial W}{\partial \xi} - W\,\frac{\partial H}{\partial \xi}\right) + \frac{2m+1}{2(\xi+\eta)}\,HW.$$

We wish now to evaluate the right hand side of (25), where we shall take for Γ the curve $\theta = 0$, which corresponds to that surface of the tapered tube which is parallel to the axis. We calculate first for any function u

$$\frac{\partial u}{\partial x} = \frac{e^{-\rho+i\theta}}{2}\left[-i\,\frac{\partial u}{\partial \rho} + \frac{\partial u}{\partial \theta}\right]$$

$$\frac{\partial u}{\partial y} = \frac{e^{-\rho-i\theta}}{2}\left[\;i\,\frac{\partial u}{\partial \rho} + \frac{\partial u}{\partial \theta}\right]$$

$$d\xi = i\,e^{\rho-i\theta}(d\rho - i\,d\theta)$$

$$d\eta = -i\,e^{\rho+i\theta}(d\rho + i\,d\theta).$$

Then we obtain from (25) and the relations following it

$$W(\rho_0,\theta_0) = i\int_{\rho_0-i\,\theta_0}^{\rho_0+i\,\theta_0} Q(\rho,0)d\rho\; +$$

$$\frac{1}{2}(HW)(\rho_0 - i\,\theta_0,0) + \frac{1}{2}(HW)(\rho_0 + i\,\theta_0,0) \qquad (26)$$

where

$$Q(\rho,\theta) = \frac{1}{2}\left[H\,\frac{\partial W}{\partial \theta} - W\,\frac{\partial H}{\partial \theta}\right] + \frac{2m+1}{C}\,e^{\rho}\,HW. \qquad (27)$$

Now (26) and (27) give an integral representation of the solution. The limits $\rho_0 - i\,\theta_0$ and $\rho_0 + i\,\theta_0$ are of course the (complex) values of ρ at which the characteristics (21a) and (21b) through (ρ_0,θ_0) intersect $\theta = 0$.

We have been proceeding formally, but it is clear that we must assume analytic boundary data, for otherwise the integrand will be path dependent.

We shall argue that the integrand $Q(\rho,0)$ is real for real values of ρ. Assume this for the moment.

If we make the change of variable

$$\rho = \rho_0 + i\phi$$

in the integrand, we get for the integral term

$$-\int_{-\theta_0}^{\theta_0} Q(\rho_0 + i\phi)\, d\phi = -\int_{-\theta_0}^{0} Q(\rho_0 + i\phi,0)d\phi$$

$$-\int_{0}^{\theta_0} Q(\rho_0 + i\phi,0)d\phi.$$

By putting $-\phi$ for ϕ in the first integral on the right hand side, we get for the integral term

$$-\int_{0}^{\theta_0}[Q(\rho + i\phi,0) + Q(\rho - i\phi,0)]d\phi,$$

which, under our assumptions, becomes

$$-2\int_{0}^{\theta_0} \mathrm{Re}Q(\rho_0 + i\phi,0)d\phi.$$

We obtain, then

$$W(\rho,\theta) =$$

$$\mathrm{Re}\left\{\int_{0}^{\theta}[W\frac{\partial H}{\partial\theta} - H\frac{\partial W}{\partial\theta} + 2\frac{HW}{C}e^{\rho+i\phi}]d\phi\right.$$

$$\left.+ H(\rho + i\theta,0)W(\rho + i\theta,0)\right\}$$

where the integrand is evaluated at $(\rho + i\phi, 0)$. This is a form of the integral representation of the solution.

Now let us look at the assumptions on Q. First of all, in order to be sure that our operations are legitimate, we must assign analytic boundary data on $W(\rho,0)$ and $\partial W/\partial\theta$ at $(\rho,0)$. These must be assigned so that they are real when ρ is real. Next look at H. There is no problem of analyticity of H. However, we shall check that H is real when θ and ρ are real, so that it will also follow that $\partial H/\partial\theta$ will be real at real $(\rho,0)$.

We have from (20) and (24) that

$$\xi + \eta = 2(c + e^{\rho}\sin\theta)$$
$$\xi_0 + \eta_0 = 2(c + e^{\rho_0}\sin\theta_0)$$
$$(\xi - \xi_0)(\eta - \eta_0) = (e^{\rho - i\theta} - e^{\rho_0 - i\theta_0})(e^{\rho + i\theta} - e^{\rho_0 + i\theta_0})$$

Now the last equation is the product of an expression with that of its complex conjugate and is thus real. From (23), it then follows that z is real for real (ρ,θ), and hence, from (22) and (23) we see that H is real for real (ρ,θ).

This establishes, at least formally, the integral representation.

7. <u>Some Polynomial and Series Solutions</u>

Although it is not possible to separate variables in the Niven coordinates, or in an equivalent coordinate system, it is possible to obtain some exact solutions of more generality than those given by Niven. To this end let us start with equation (3) for the reduced potential $U^{(m)}$ in cylindrical coordinates. We drop the superscript m for notational convenience. We have, then

$$R^2\nabla^2U + (\tfrac{1}{4} - m^2)U = 0. \tag{28}$$

We substitute for U the series

$$U = \sum_{k=0}^{\infty} R^{p+k}\ f_k(Z) \tag{29}$$

where the index p and the functions $f_k(Z)$ are to be determined.

Substitution of (28) into (29) yields

$$[p(p-1) + \tfrac{1}{4} - m^2]f_0(Z) + [(p+1)p + \tfrac{1}{4} - m^2]f_1(Z)R$$

$$+ \sum_{k=2}^{\infty}\left\{[(p+k)(p+k-1) + \tfrac{1}{4} - m^2]f_k(Z) + f''_{k-2}(Z)\right\} R^k = 0. \tag{30}$$

Setting the coefficient of $f_0(Z)$ to zero, we get the indicial equation

$$(p - \tfrac{1}{2})^2 - m^2 = 0$$

or

$$p = \tfrac{1}{2} \pm m.$$

We choose the larger root $p = \frac{1}{2} + m$ of the indicial equation. (As with Bessel functions, the smaller root will give nothing new.) We obtain by substitution into (30)

$$(2m+2)f_1(Z) = 0 \tag{31}$$

$$k(2m+k)f_k(Z) + f''_{k-2}(Z) = 0. \tag{32}$$

Equations (31) and (32) taken together tell us that

$$f_k(Z) = 0 \quad \text{for odd } k$$

and that for even k, k = 2j, we have

$$f_{2j} = \frac{f''_{2(j-1)}}{2^2 j(j+m)} \ . \tag{33}$$

This gives us a method of obtaining solutions, since we can now pick $f_0(\circledast)$ as we wish and make use of the recursion relation (32). We shall explore some of the consequences of this procedure.

We see, in particular, from (33) that if we choose f_0 to be a polynomial in Z, then the number of terms in the expansion (29) will be finite. The solution will be a square root of R times a polynomial in R and Z. We shall investigate the case

$$f_0(Z) = Z^n, \quad n = 0,1,2,\ldots \tag{34}$$

From these one can, in fact, by addition and multiplication by constants, arrive at the results for any choice of a polynomial for f_0.

Now (34) and (33) tell us that

$$f_{2j} = c_j \ x^{n-2j} \tag{35}$$

for some c_j to be determined. If we put (35) into (33) we obtain

$$c_j = - \frac{(n-2j+2)(n-2j+1)}{2^2 \ j \ (j+m)} \ c_{j-1}. \tag{36}$$

The solution to the recursion relation (36) is

$$c_j = (-1)^j \ \frac{n!m!}{2^{2j} \ j!(j+m)!(n-2j)!} \ c_0 \tag{37}$$

where $j = 0,1,\ldots,n/2$ if n is even and $j = 0,1,\ldots,(n-1)/2$ if n is odd.

If we take $c_0 n!m! = 1$, we obtain then a solution

$$U = R^{\frac{1}{2}+m} \sum_{j=0}^{h(n)} \frac{(-1)^j}{2^{2j}(n-2j)!\,j!\,(m+j)!} R^{2j} Z^{n-2j} \tag{38}$$

where $h(n) = n/2$ if n is even and $h(n) = (n-1)/2$ if n is odd.

Equation (38) gives, then, a square root of R times a polynomial in R and Z. Now returning to Niven coordinates, we obtain

$$U = (c+r\sin\theta)^{m+\frac{1}{2}}(r\cos\theta)^n \sum_{j=0}^{h(n)} \frac{(-1)^j}{2^{2j}(n-2j)!\,j!\,(n+j)!}\left(\frac{c}{r}\sec\theta+\tan\theta\right)^{2j}.$$

Since there are finitely many terms, there is no problem of convergence.

Another set of exact solutions can be generated by writing

$$f_0(Z) = Z^{-n}$$

where n is an integer. We obtain, then, by the same process

$$U = R^{\frac{1}{2}+m} Z^{-n} \sum_{j=0}^{\infty} \frac{(-1)^j\,(n+2j-1)!}{2^{2j}\,j!\,(j+m)!} \left(\frac{R}{Z}\right)^{2j}.$$

According to the ratio test, this series will converge for $|\frac{R}{Z}| < 1$.

In the Niven coordinates, then we obtain

$$U = r^{-n}(c+r\sin\theta)^{\frac{1}{2}+m} \sum_{j=0}^{\infty} \frac{(-1)^m(n+2j-1)!}{2^{2j}\,j!\,(j+m)!}\left(\frac{c}{r}\sec\theta+\tan\theta\right)^{2j}$$

which converges when

$$c+r\sec\theta < r\cos\theta \quad \text{(1st quadrant)}$$

or $\qquad r > \frac{c}{\cos\theta - \sin\theta}$.

Now let us consider a taper angle $\alpha < \frac{\pi}{2}$, so that we are concerned with, say,

$$0 \leqslant \theta \leqslant \alpha.$$

Then $\cos\theta - \sin\theta$ is decreasing with θ, since

$$\frac{d}{d\theta}(\cos\theta - \sin\theta) = -\sin\theta - \cos\theta.$$

Thus

$$\frac{c}{\cos\theta - \sin\theta} \leqslant \frac{c}{\cos\alpha - \sin\alpha}, \quad 0 \leqslant \theta \leqslant \alpha.$$

Therefore convergence in the wedge region will occur for

$$r > \frac{c}{\cos\alpha - \sin\alpha}$$

which means that the solution will be valid away from the singularity where the outer and inner surface of the tapered tube meet. This is the sort of region that figures in a gun tube.

For the case of $m = 0$, we have also succeeded in finding an additional solution involving a logarithm and a polynomial. We seek solutions to (28) in the form

$$U = V \ln R + \sum_{j=0}^{\infty} b_j R^{\frac{1}{2}+2j} Z^{n-2j} \tag{39}$$

where V satisfies (28) with $m = 0$. We obtain

$$0 = R\nabla^2 U + \frac{U}{4} = [R^2\nabla^2 U + \frac{U}{4}] \ln R$$

$$+ 2R\frac{\partial V}{\partial R} - V + \sum_{j=0}^{\infty} (4j^2 - \tfrac{1}{4}) b_j \cdot R^{\frac{1}{2}+2j} Z^{n-2j} .$$

We take for V the solution (38) which we write as

$$V = R^{\frac{1}{2}} \sum_{j=0}^{h(n)} c_j \, R^{2j} \, Z^{n-2j},$$

where c_j is given by (37). After some straightforward manipulation we obtain the recursion relation

$$b_j = -\frac{1}{4j^2}[(n-2j+2)(n-2j+1)b_{j-1} + 4jc_j], \quad j = 1,2,\ldots \tag{40}$$

We note that since there are only a finite number of non-zero c_j, and since the coefficient of b_{j-1} in (40) becomes zero for $j = (n+2)/2$ or $j = (n+1)/2$ (depending on whether n is even or odd) there are only a finite number of non-zero b_j and so both V and the summation term in (39) become polynomials.

These solutions, then, in Niven coordinates become

$$r^{(n+\frac{1}{2})}\sqrt{\frac{c}{r} + \sin\theta}\left[\sum_{j=0}^{h(n+2)} [\tfrac{c}{r}\sec\theta + \tan\theta]^{2j}[c_j \ell n(c + r\sin\theta) + b_j]\right].$$

Some thought must be given to the best way to assemble these solutions in order to satisfy given boundary conditions.

8. Short Annotated Bibliography

The annotations here will also be brief since the information that they convey is also in the text, often in more detail.

[1] Snow, C. Hypergeometric and Legendre Functions with Application to Integral Equations of Potential Theory, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 19 (1952).

This work seems very rich and may well hold the clue to a procedure for attacking the tapered tube problem. It does establish criteria for whether one may separate variables for Laplace's equation after transforming the polar coordinates R, Z to another set by a conformal mapping. However, the relevant chapter which we feel could still contain some clues is Chapter IX, " Some Integral Equations of Potential Theory". We advocate some further study to see if the method can be adapted to the tapered tube. However, it may take some clever insight to implement it.

[2] Heins, A. E. " Axially-Symmetric Boundary Value Problems" Bull. Amer. Math. Soc. 71, pp. 787-808 (1965).

This is a review of the use of the Poisson Integral formulation to solve problems of potential theory by the means of integral equations. A feature of the method is that the solution is represented by an integral whose integrand involves the values of the solution on the Z axis. Another relation, such as the Helmholtz representation, for example, is then used to set up a relation, actually an integral equation, with which to determine the unknown integrand in the Poisson integral. When this relation can be solved, then one has the solution to the problem represented as an integral. Problems given as examples include the electrostatic fields about discs and circular lenses with axes on the Z axis.

It is not clear at the time of writing of this report whether or not the method can be effectively used for the tapered tube since 1) the method is basically designed for functions which are regular on the Z axis (or at least a portion of it) and 2) its application seems to depend on its proper use in an appropriate coordinate system. Whether and how to use it in whole or in part for the tapered tube problem deserves further study.

[3] Zak, A. R. "Elastic Analysis of Cylindrical Configurations with Stress Singularities", J. Applied Mech. 39 (E) pp. 501-506 (1972).

As the title implies, the author is interested here in a study of stress singularities in a cylindrical tube. However, the coordinate system which he adopts is essentially the same as the Niven coordinates. The method does not use a sequence of exact solutions, but it does obtain a series which converges to an exact solution where it converges. As the method is applied, it is limited by a singular curve ($\rho \sin\phi = 1$), a function which occurs in the equations $(1 - \rho \sin\phi)^{-1}$. As the distance ρ from $\rho = 0$ (which would be the intersection of the inner and outer surfaces in the tapered tube problem) gets larger, the location of the singular curve gets closer and closer to where $\sin\phi = 0$, which limits the region in which Zak's method could be used, at least without some modification. We calculated that for a taper angle of 2° one is limited to a distance less than about 28 times the distance from the axis to the intersection of the inner and outer surfaces of the tapered tube.

[4] Niven, D. W. " On a Special Form of Laplace's Equation" Messenger of Mathematics X, pp. 114-117 (1881).

Except for the paper by Riemann [7], this work of Niven is the only one we found which treated the coordinate system appropriate to the tapered tube. Some special solutions were found, but do not apply to the tapered tube.

[5] Temple, G. " Whittaker's Work in the Integral Representations of Harmonic Functions" Proc. Edinburgh Math. Soc. 11, pp. 11-24 (1958).

This is a nice review article which goes into the history of integral representations of potential functions. It contains a derivation of the Poisson Integral and developments leading to the Whittaker integral, as well as comments on its applicability.

[6] Whittaker, E. T. " On the Partial Differential Equations of Mathematical Physics" Mathematische Annalen 57, pp. 333-355 (1903).

This paper is of interest because of the development of the Whittaker integral and a demonstration of its use in an example problem (the potential of a prolate spheroid).

[7] Riemann, B. " Ueber das Potential eines Ringes", Gesammelte Werke, Chapter XXIV.

This is of interest because coordinates appropriate to the tapered tube are used. However, the method of solution of a problem, namely that of an anchor ring, to which the

paper is devoted, does not in any clear way indicate how to attack the tapered tube.

[8] Weinstein, A. " Generalized Axially Symmetric Potential Theory", Bull. Amer. Math. Soc. 59, pp. 20-38 (1953).

Weinstein develops a method of attacking problems by use of a generalization of potential theory. This subject merits some further study to see if some relation can be found to the tapered tube problem.

In addition we mention some papers of W. D. Collins, who uses the method described by Heins. These are of interest for the detail which they provide on the method.

[9] Collins, W. D. " On the Solution of Some Axisymmetric Boundary Value Problems by Means of Integral Equations I, Some Problems for a Spherical Cap" , Quart. J. Mech. Appl. Math 12, pp. 232-241 (1959).

[10] Collins, W. D. " On the Solution of Some Axisymmetric Boundary Value Problems by Means of Integral Equations II, Further Problems for a Circular Disc and a Spherical Cap", Mathematika 6, pp. 120-133 (1959).

[11] Collins, W. D. " On the Solution of Some Axisymmetric Boundary Value Problems by Means of Integral Equations IV, The Electrostatic Potential Due to a Spherical Cap Between Two Infinite Conducting Planes", Proc. Edinburgh Math. Soc. (2) 12, pp. 95-106 (1960/61).

[12] Collins, W. D. " On the Solution of Some Axisymmetric Boundary Value Problems by Means of Integral Equations VII, The Electrostatic Potential Due to a Spherical Cap Situated Inside a Circular Cylinder", Proc. Edinburgh Math. Soc. (2) 13, pp. 13-23 (1962).

[13] Green and Zerna, " Theoretical Elasticity" , Oxford U. Press, Oxford (1968).

[14] Allen, D. N. " Relaxation Methods", McGraw-Hill, New York (1954).

[15] Sommerfeld, A. " Partial Differential Equations in Physics", Academic Press, New York (1949).

9. Conclusion

The problem of the best way to proceed to obtain analytical solutions of the tapered tube problem remains unresolved at the moment. We feel that some progress has been made in developing exact solutions in the appropriate coordinate system. Perhaps a method of assembling these solutions to fit boundary data would be fruitful. Perhaps a formulation in integral equation form, would be the way to proceed. Again we feel that some progress has been made by our integral representation of section 6. And of course Zak's basic idea could be developed further. Indeed, for small taper angles it could be useful not too far from the intersection of the inner and outer surfaces as it stands. But it would be better to try to develop the idea of the method for points far from this intersection. We have, indeed, made a stab at the problem and hope that our contributions will be the first step in the solution. However, at this point, we feel that much remains to be done.

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